Solution of Simultaneous Equation System:

It is already mentioned in the previous section that matrix algebra is generally used to solve a system of simultaneous linear equations. There are basically two methods of finding solutions of a simultaneous linear equation systems –

(a) Matrix Inversion Method, and

(b) Cramer's Rule

(a) Matrix Inversion Method:

Let us consider a general form of a linear system of 'n' simultaneous equations with 'n' endogenous variables $x_1, x_2, x_3, ---, x_n$ such that

$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + ---- + a_{1n}x_n = C_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + ---- + a_{2n}x_n = C_2 \\ ---------- \\ a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + ---- + a_{nn}x_n = C_n \end{array}\right\} \quad \text{--- (1)}$$

Now converting the above simultaneous equation system into matrix form or notation, we have

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & --- & a_{1n} \\ a_{21} & a_{22} & a_{23} & --- & a_{2n} \\ --- & --- & --- & --- & --- \\ a_{n1} & a_{n2} & a_{n3} & --- & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ -- \\ x_n \end{bmatrix} = \begin{bmatrix} C_1 \\ C_2 \\ -- \\ C_n \end{bmatrix} \quad \text{--- (2)}$$

Again it is assumed that

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}, \quad X = \begin{bmatrix} x_1 \\ x_2 \\ \cdots \\ x_n \end{bmatrix} \text{ and } C = \begin{bmatrix} c_1 \\ c_2 \\ \cdots \\ c_n \end{bmatrix}$$

$\therefore$ The equation (2) takes the following form

$$AX = C$$

$\Rightarrow A^{-1}.A.X = A^{-1}.C$ (multiplying both sides by A^{-1})

$\Rightarrow I.X = A^{-1}.C$ ($\because A^{-1}.A = I$)

$\Rightarrow X = A^{-1}.C$ ——— (3)

The above general procedure is followed to find out the ~~good~~ solution of simultaneous linear equation system through matrix inversion method.

Example:

1. Solve the following equation system by matrix inversion

$$3x + 3y = 5$$
$$3x - 4y = 2$$

Solution:

The given equation system is

$$\left.\begin{aligned} 3x + 3y &= 5 \\ 3x - 4y &= 2 \end{aligned}\right\} \text{——— (1)}$$

Now, converting the above equation system into matrix form or notation we have

$$\underbrace{\begin{bmatrix} 3 & 3 \\ 3 & -4 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x \\ y \end{bmatrix}}_{X} = \underbrace{\begin{bmatrix} 5 \\ 2 \end{bmatrix}}_{C}$$

or $AX = C$

or $X = A^{-1}.C$ ——— (2)

We know that $A^{-1} = \dfrac{Adj(A)}{|A|}$

and $Adj(A) = [\text{Cofactor matrix of } A]'$

Now,

$$|A| = \begin{vmatrix} 3 & 3 \\ 3 & -4 \end{vmatrix} = (3.(-4) - 3.3) = (-12 - 9) = -21$$

Now, Co-factor matrix of $A = \begin{bmatrix} (-1)^{1+1}(-4) & (-1)^{1+2}.(3) \\ (-1)^{2+1}.(3) & (-1)^{2+2}(3) \end{bmatrix}_{(2\times 2)}$

$$= \begin{bmatrix} -4 & -3 \\ -3 & 3 \end{bmatrix}$$

$$\therefore \text{Adj of } A = \begin{bmatrix} -4 & -3 \\ -3 & 3 \end{bmatrix}' = \begin{bmatrix} -4 & -3 \\ -3 & 3 \end{bmatrix}$$

Putting the values of $|A|$, $Adj(A)$ and c in equation (2), we have

$$X = -\frac{1}{21}\begin{bmatrix} -4 & -3 \\ -3 & 3 \end{bmatrix}\begin{bmatrix} 5 \\ 2 \end{bmatrix}$$

$$= -\frac{1}{21}\begin{bmatrix} ((-4).5 + (-3).2) \\ ((-3).5 + 3.2) \end{bmatrix}$$

$$= -\frac{1}{21}\begin{bmatrix} (-9 - 6) \\ (-15 + 6) \end{bmatrix}$$

$$= -\frac{1}{21}\begin{bmatrix} -15 \\ -9 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{15}{21} \\ \frac{9}{21} \end{bmatrix}$$

$\therefore x = 15/21$ and $y = \frac{9}{21}$

2. Solve the following equation system by matrix inversion method

$$2x_1 + 2x_2 = 16$$
$$4x_1 - 3x_2 = 11$$

Solution: The given equation system is

$$\left.\begin{aligned} 2x_1 + 2x_2 &= 16 \\ 4x_1 - 3x_2 &= 11 \end{aligned}\right\} \text{— (1)}$$

Now converting the above equation system into matrix form or notation, we have

$$\underbrace{\begin{bmatrix} 2 & 2 \\ 4 & -3 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{X} = \underbrace{\begin{bmatrix} 16 \\ 11 \end{bmatrix}}_{C}$$

or $AX = C$

or $X = A^{-1} \cdot C$ ——— (2)

We know that $A^{-1} = \dfrac{Adj(A)}{|A|}$

and $Adj(A) = [\text{co-factor matrix of } A]'$

$\therefore |A| = \begin{vmatrix} 2 & 2 \\ 4 & -3 \end{vmatrix} = (2.(-3) - 2.4) = (-6-8) = -14$

Now, co-factor matrix of $A = \begin{bmatrix} (-1)^{1+1}.(-3) & (-1)^{1+2}.(4) \\ (-1)^{2+1}.(2) & (-1)^{2+2}.(2) \end{bmatrix}_{(2\times 2)}$

$$= \begin{bmatrix} -3 & -4 \\ -2 & 2 \end{bmatrix}_{(2\times 2)}$$

$\therefore Adj(A) = \begin{bmatrix} -3 & -4 \\ -2 & 2 \end{bmatrix}' = \begin{bmatrix} -3 & -2 \\ -4 & 2 \end{bmatrix}$

Now putting the values of $|A|$, $Adj(A)$ and C in equation (2), we have

$$X = -\frac{1}{14}\begin{bmatrix} -3 & -2 \\ -4 & 2 \end{bmatrix}\begin{bmatrix} 16 \\ 11 \end{bmatrix}$$

$$\Rightarrow X = -\frac{1}{14}\begin{bmatrix} (-3).16 + (-2).11) \\ (-4).16 + 2.11) \end{bmatrix}$$

$$= -\frac{1}{14}\begin{bmatrix} -48 - 22 \\ -64 + 22 \end{bmatrix}$$

$$= -\frac{1}{14}\begin{bmatrix} -70 \\ -42 \end{bmatrix} = \begin{bmatrix} \frac{-70}{-14} \\ \frac{-42}{-14} \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$$

$\therefore x_1 = 5$ and $x_2 = 3$

3. Solve the following equation system by matrix inversion method:

$$3x + 2y = 13$$
$$9x - 3y = 21$$

Solution: The given equation system is

$$\left.\begin{aligned} 3x + 2y &= 13 \\ 9x - 3y &= 21 \end{aligned}\right\} \text{—— ①}$$

Now converting the above equation system into matrix form, we have

$$\underbrace{\begin{bmatrix} 3 & 2 \\ 9 & -3 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x \\ y \end{bmatrix}}_{X} = \underbrace{\begin{bmatrix} 13 \\ 21 \end{bmatrix}}_{C}$$

or $AX = C$

or $X = A^{-1} \cdot C$ —— ②

We know that $A^{-1} = \dfrac{Adj(A)}{|A|}$,

and $Adj(A) = [\text{Co-factor Matrix of } A]'$

$$\therefore |A| = \begin{vmatrix} 3 & 2 \\ 9 & -3 \end{vmatrix} = (3.(-3) - 2.9) = (-9 - 18) = -27$$

Now, Co-factor matrix of $A = \begin{bmatrix} (-1)^{1+1}.(-3) & (-1)^{1+2}.(9) \\ (-1)^{2+1}.(2) & (-1)^{2+2}(3) \end{bmatrix}_{(2\times2)}$

$$= \begin{bmatrix} -3 & -9 \\ -2 & 3 \end{bmatrix}_{(2\times2)}$$

$$\therefore Adj(A) = \begin{bmatrix} -3 & -9 \\ -2 & 3 \end{bmatrix}' = \begin{bmatrix} -3 & -2 \\ -9 & 3 \end{bmatrix}$$

Now, putting the values of $|A|$, $Adj(A)$ and C in equation ②, we have

$$X = -\frac{1}{27} \begin{bmatrix} -3 & -2 \\ -9 & 3 \end{bmatrix} \begin{bmatrix} 13 \\ 21 \end{bmatrix}$$

$$\text{or, } X = -\frac{1}{27}\begin{bmatrix}(-3).13 + (-2).21\\ (-9).13 + 3.21\end{bmatrix}$$

$$= -\frac{1}{27}\begin{bmatrix}(-39-42)\\ (-117+63)\end{bmatrix}$$

$$= -\frac{1}{27}\begin{bmatrix}-81\\ -54\end{bmatrix}$$

$$= \begin{bmatrix}\frac{-81}{-27}\\ \frac{-54}{-27}\end{bmatrix} = \begin{bmatrix}3\\ 2\end{bmatrix}$$

$\therefore x = 3$ and $y = 2$

4. Solve the following equation system by matrix inversion method.

$$2a + 2b + c = 9$$
$$a + 3b + 2c = 13$$
$$2a - b + 3c = 9$$

Solution: The given equation system is

$$\left.\begin{aligned} 2a + 2b + c &= 9\\ a + 3b + 2c &= 13\\ 2a - b + 3c &= 9 \end{aligned}\right\} \text{---} (1)$$

The above equation system is converted into matrix form or notation & we have

$$\underbrace{\begin{bmatrix}2 & 2 & 1\\ 1 & 3 & 2\\ 2 & -1 & 3\end{bmatrix}}_{A}\underbrace{\begin{bmatrix}a\\ b\\ c\end{bmatrix}}_{X} = \underbrace{\begin{bmatrix}9\\ 13\\ 9\end{bmatrix}}_{C}$$

or $AX = C$

or $X = A^{-1}.C$ ---------- (2)

We know that, $A^{-1} = \frac{Adj(A)}{|A|}$

and $Adj(A) = [\text{co-factor matrix of } A]'$

$$\therefore |A| = \begin{vmatrix}2 & 2 & 1\\ 1 & 3 & 2\\ 2 & -1 & 3\end{vmatrix} = 2\begin{vmatrix}3 & 2\\ -1 & 3\end{vmatrix} - 2\begin{vmatrix}1 & 2\\ 2 & 3\end{vmatrix} + 1\begin{vmatrix}1 & 3\\ 2 & -1\end{vmatrix}$$

$$= 2(3.3 - 2.(-1)) - 2(1.3 - 2.2) + 1(1.(-1) - 3.2)$$

$$= 2(9+2) - 2(3-4) + 1(-1-6)$$

$$= (2\times 11) - (2\times(-1)) + (1\times(-7)) = (22 + 2 - 7) = 17$$

Co-factor matrix of $A = \begin{bmatrix} (-1)^{1+1}\begin{vmatrix}3&2\\-1&3\end{vmatrix} & (-1)^{1+2}\begin{vmatrix}1&2\\2&3\end{vmatrix} & (-1)^{1+3}\begin{vmatrix}1&3\\2&-1\end{vmatrix} \\ (-1)^{2+1}\begin{vmatrix}2&1\\-1&3\end{vmatrix} & (-1)^{2+2}\begin{vmatrix}2&1\\2&3\end{vmatrix} & (-1)^{2+3}\begin{vmatrix}2&2\\2&-1\end{vmatrix} \\ (-1)^{3+1}\begin{vmatrix}2&1\\3&2\end{vmatrix} & (-1)^{3+2}\begin{vmatrix}2&1\\1&2\end{vmatrix} & (-1)^{3+3}\begin{vmatrix}2&2\\1&3\end{vmatrix} \end{bmatrix}$

$= \begin{bmatrix} +(3.3-2.(-1)) & -(1.3-2.2) & +(1.(-1)-3.2) \\ -(2.3-1.(-1)) & +(2.3-1.2) & -(2.(-1)-2.2) \\ +(2.2-1.3) & -(2.2-1.1) & +(2.3-2.1) \end{bmatrix}_{(3\times3)}$

$= \begin{bmatrix} +(9+2) & -(3-4) & +(-1-6) \\ -(6+1) & +(6-2) & -(-2-4) \\ +(4-3) & -(4-1) & +(6-2) \end{bmatrix}_{(3\times3)}$

$= \begin{bmatrix} 11 & 1 & -7 \\ -7 & 4 & 6 \\ 1 & -3 & 4 \end{bmatrix}_{(3\times3)}$

$\therefore Adj(A) = \begin{bmatrix} 11 & 1 & -7 \\ -7 & 4 & 6 \\ 1 & -3 & 4 \end{bmatrix}' = \begin{bmatrix} 11 & -7 & 1 \\ 1 & 4 & -3 \\ -7 & 6 & 4 \end{bmatrix}$

Now putting the values of $|A|$, $Adj(A)$ and C in equation (2), we have

$$X = \frac{1}{17}\begin{bmatrix} 11 & -7 & 1 \\ 1 & 4 & -3 \\ -7 & 6 & 4 \end{bmatrix}\begin{bmatrix} 9 \\ 13 \\ 9 \end{bmatrix}$$

$$= \frac{1}{17}\begin{bmatrix} (11.9+(-7).13+1.9) \\ (1.9+4.13+(-3).9) \\ ((-7).9+6.13+4.9) \end{bmatrix}$$

$$= \frac{1}{17}\begin{bmatrix} (99-91+9) \\ (9+52-27) \\ (-63+78+36) \end{bmatrix}$$

$$= \frac{1}{17}\begin{bmatrix} 17 \\ 34 \\ 51 \end{bmatrix} = \begin{bmatrix} 17/17 \\ 34/17 \\ 51/17 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$\therefore a = 1,\ b = 2$ and $c = 3$

5. Solve the following equation system by matrix inversion method.

$$x_1 - 2x_2 + 3x_3 = 1$$
$$2x_1 + x_2 - 2x_3 = -1$$
$$3x_1 - x_2 + 4x_3 = 3$$

Solution: The given equation system is

$$\left.\begin{aligned} x_1 - 2x_2 + 3x_3 &= 1 \\ 2x_1 + x_2 - 2x_3 &= -1 \\ 3x_1 - x_2 + 4x_3 &= 3 \end{aligned}\right\} \quad \text{---- (1)}$$

Converting the above equation system into matrix form or notation, we have

$$\underbrace{\begin{bmatrix} 1 & -2 & 3 \\ 2 & 1 & -2 \\ 3 & -1 & 4 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{X} = \underbrace{\begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}}_{C}$$

or $AX = C$

or $X = A^{-1}C$ ---------- (2)

We know that $A^{-1} = \dfrac{Adj(A)}{|A|}$,

and $Adj(A) = [\text{Co-factor matrix of } A]'$

$$\therefore |A| = \begin{vmatrix} 1 & -2 & 3 \\ 2 & 1 & -2 \\ 3 & -1 & 4 \end{vmatrix} = 1\begin{vmatrix} 1 & -2 \\ -1 & 4 \end{vmatrix} - (-2)\begin{vmatrix} 2 & -2 \\ 3 & 4 \end{vmatrix} + 3\begin{vmatrix} 2 & 1 \\ 3 & -1 \end{vmatrix}$$

$$= 1(1.4 - (-2).(-1)) + 2(2.4 - (-2).3) + 3(2.(-1) - 1.3)$$

$$= 1(4-2) + 2(8+6) + 3(-2-3)$$

$$= (1 \times 2) + (2 \times 14) + (3 \times (-5))$$

$$= (2 + 28 - 15) = 15$$

Co-factor matrix of $A = \begin{bmatrix} (-1)^{1+1}\begin{vmatrix} 1 & -2 \\ -1 & 4 \end{vmatrix} & (-1)^{1+2}\begin{vmatrix} 2 & -2 \\ 3 & 4 \end{vmatrix} & (-1)^{1+3}\begin{vmatrix} 2 & 1 \\ 3 & -1 \end{vmatrix} \\ (-1)^{2+1}\begin{vmatrix} -2 & 3 \\ -1 & 4 \end{vmatrix} & (-1)^{2+2}\begin{vmatrix} 1 & 3 \\ 3 & 4 \end{vmatrix} & (-1)^{2+3}\begin{vmatrix} 1 & -2 \\ 3 & -1 \end{vmatrix} \\ (-1)^{3+1}\begin{vmatrix} -2 & 3 \\ 1 & -2 \end{vmatrix} & (-1)^{3+2}\begin{vmatrix} 1 & 3 \\ 2 & -2 \end{vmatrix} & (-1)^{3+3}\begin{vmatrix} 1 & -2 \\ 2 & 1 \end{vmatrix} \end{bmatrix}_{(3\times 3)}$$

$$= \begin{bmatrix} +(1.4-(-2).(-1)) & -(2.4-(-2).3) & +(2.(-1)-1.3) \\ -((-2).4-3.(-1)) & +(1.4-3.3) & -(1.(-1)-(-2).3) \\ +((-2).(-2)-3.1) & -(1.(-2)-3.2) & +(1.1-(-2).2) \end{bmatrix}_{(3\times3)}$$

$$= \begin{bmatrix} (4-2) & -(8+6) & (-2-3) \\ -(-8+3) & (4-9) & -(-1+6) \\ (4-3) & -(-2-6) & (1+4) \end{bmatrix}_{(3\times3)}$$

$$= \begin{bmatrix} 2 & -14 & -5 \\ 5 & -5 & -5 \\ 1 & 8 & 5 \end{bmatrix}_{(3\times3)}$$

Now, $Adj(A) = \begin{bmatrix} 2 & -14 & -5 \\ 5 & -5 & -5 \\ 1 & 8 & 5 \end{bmatrix}'_{(3\times3)} = \begin{bmatrix} 2 & 5 & 1 \\ -14 & -5 & 8 \\ -5 & -5 & 5 \end{bmatrix}_{(3\times3)}$

Putting the values of $|A|$, $Adj(A)$ and C in eq^n. (2), we have -

$$\therefore A^{-1} = \frac{1}{15}\begin{bmatrix} 2 & 5 & 1 \\ -14 & -5 & 8 \\ -5 & -5 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}$$

$$= \frac{1}{15}\begin{bmatrix} (2.1+5.(-1)+1.3) \\ (-14).1+(-5).(-1)+8.3) \\ (-5).1+(-5).(-1)+5.3) \end{bmatrix}$$

$$= \frac{1}{15}\begin{bmatrix} (2-5+3) \\ (-14+5+24) \\ (-5+5+15) \end{bmatrix}$$

$$= \frac{1}{15}\begin{bmatrix} 0 \\ 15 \\ 15 \end{bmatrix} = \begin{bmatrix} 0/15 \\ 15/15 \\ 15/15 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

$\therefore x_1 = 0$, $x_2 = 1$ and $x_3 = 1$

6. Solve the following equation system by matrix inversion method:

$$4x + 2y - z = 40$$
$$x + 3z = 38$$
$$2x + 3y = 43$$

Solution: The given equation system is

$$4x + 2y - z = 40$$
$$x + 3z = 38$$
$$2x + 3y = 43$$

The given equation system can be rewritten as -

$$\left.\begin{aligned} 4x + 2y - 1z &= 40 \\ 1x + 0y + 3z &= 38 \\ 2x + 3y + 0z &= 43 \end{aligned}\right\} \quad \text{——— (1)}$$

Converting the above equation system into matrix form or notation as -

$$\underbrace{\begin{bmatrix} 4 & 2 & -1 \\ 1 & 0 & 3 \\ 2 & 3 & 0 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x \\ y \\ z \end{bmatrix}}_{X} = \underbrace{\begin{bmatrix} 40 \\ 38 \\ 43 \end{bmatrix}}_{C}$$

or $AX = C$

$\Rightarrow X = A^{-1}C$ ——— (2)

We know that $A^{-1} = \frac{Adj(A)}{|A|}$,

and $Adj(A) = [\text{co-factor matrix of } A]'$

$$\therefore |A| = \begin{vmatrix} 4 & 2 & -1 \\ 1 & 0 & 3 \\ 2 & 3 & 0 \end{vmatrix} = 4\begin{vmatrix} 0 & 3 \\ 3 & 0 \end{vmatrix} - 2\begin{vmatrix} 1 & 3 \\ 2 & 0 \end{vmatrix} + (-1)\begin{vmatrix} 1 & 0 \\ 2 & 3 \end{vmatrix}$$

$$= 4(0.0 - 3.3) - 2(1.0 - 3.2) - 1(1.3 - 0.2)$$
$$= 4(0 - 9) - 2(0 - 6) - 1(3 - 0)$$
$$= (4 \times (-9)) - (2 \times (-6)) - (1 \times 3)$$
$$= (-36 + 12 - 3)$$
$$= -27$$

Co-factor matrix of $A = \begin{bmatrix} (-1)^{1+1}\begin{vmatrix}0&3\\3&0\end{vmatrix} & (-1)^{1+2}\begin{vmatrix}1&3\\2&0\end{vmatrix} & (-1)^{1+3}\begin{vmatrix}1&0\\2&3\end{vmatrix} \\ (-1)^{2+1}\begin{vmatrix}2&-1\\3&0\end{vmatrix} & (-1)^{2+2}\begin{vmatrix}4&-1\\2&0\end{vmatrix} & (-1)^{2+3}\begin{vmatrix}4&2\\2&3\end{vmatrix} \\ (-1)^{3+1}\begin{vmatrix}2&-1\\0&3\end{vmatrix} & (-1)^{3+2}\begin{vmatrix}4&-1\\1&3\end{vmatrix} & (-1)^{3+3}\begin{vmatrix}4&2\\1&0\end{vmatrix} \end{bmatrix}$

$$= \begin{bmatrix} +(0.0-3.3) & -(1.0-3.2) & +(1.3-0.2) \\ -(2.0-(-1).3) & +(4.0-(-1).2) & -(4.3-2.2) \\ +(2.3-(-1).0) & -(4.3-(-1).1) & +(4.0-2.1) \end{bmatrix}_{(3\times3)}$$

$$= \begin{bmatrix} (0-9) & -(0-6) & (3-0) \\ -(0+3) & (0+2) & -(12-4) \\ (6+0) & -(12+1) & (8-2) \end{bmatrix}_{(3\times3)}$$

$$= \begin{bmatrix} -9 & 6 & 3 \\ -3 & 2 & -8 \\ 6 & -13 & -2 \end{bmatrix}_{(3\times3)}$$

$$\therefore Adj(A) = \begin{bmatrix} -9 & 6 & 3 \\ -3 & 2 & -8 \\ 6 & -13 & -2 \end{bmatrix}'_{(3\times3)} = \begin{bmatrix} -9 & -3 & 6 \\ 6 & 2 & -13 \\ 3 & -8 & -2 \end{bmatrix}_{(3\times3)}$$

Now, putting the values of $|A|$, $Adj(A)$ and C in equation (2), we have -

$$X = \frac{1}{-27}\begin{bmatrix} -9 & -3 & 6 \\ 6 & 2 & -13 \\ 3 & -8 & -2 \end{bmatrix}\begin{bmatrix} 40 \\ 38 \\ 43 \end{bmatrix}$$

$$= \frac{1}{-27}\begin{bmatrix} (-9).40+(-3).38+6.43 \\ (6.40+2.38+(-13).43) \\ (3.40+(-8).38+(-2).43) \end{bmatrix}$$

$$= \frac{1}{-27}\begin{bmatrix} (-360-114+258) \\ (240+76-559) \\ (120-304-86) \end{bmatrix}$$

$$= \frac{1}{-27}\begin{bmatrix} -216 \\ -243 \\ -270 \end{bmatrix} = \begin{bmatrix} \frac{-216}{-27} \\ \frac{-243}{-27} \\ \frac{-270}{-27} \end{bmatrix} = \begin{bmatrix} 8 \\ 9 \\ 10 \end{bmatrix}$$

$\therefore x = 8,\ y = 9$ and $z = 10$

7. Solve the equation system by matrix inversion method -

$$x_1 + x_2 + x_3 = 6$$
$$2x_1 + x_2 - x_3 = 1$$
$$x_1 - x_2 + x_3 = 2$$

Solution: The given equation system is

$$\left.\begin{aligned} x_1 + x_2 + x_3 &= 6 \\ 2x_1 + x_2 - x_3 &= 1 \\ x_1 - x_2 + x_3 &= 2 \end{aligned}\right\} \quad \text{——— (1)}$$

converting the above equation system into matrix form or notation, we have

$$\underbrace{\begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}}_{A}\underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{X} = \underbrace{\begin{bmatrix} 6 \\ 1 \\ 2 \end{bmatrix}}_{C}$$

or $AX = C$

or $X = A^{-1} \cdot C$ ——— (2)

We know that $A^{-1} = \frac{Adj(A)}{|A|}$, and

$$Adj(A) = [\text{co-factor matrix of } A]'$$

$$\therefore |A| = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{vmatrix} = 1\begin{vmatrix} 1 & -1 \\ -1 & 1 \end{vmatrix} - 1\begin{vmatrix} 2 & -1 \\ 1 & 1 \end{vmatrix} + 1\begin{vmatrix} 2 & 1 \\ 1 & -1 \end{vmatrix}$$

$$= 1(1.1 - (-1)(-1)) - 1(2.1 - (-1).1) + 1(2.(-1) - 1.1)$$
$$= 1(1-1) - 1(2+1) + 1(-2-1)$$
$$= (1 \times 0) - (1 \times 3) + (1 \times (-3))$$
$$= (0 - 3 - 3) = -6$$

Co-factor matrix of $A = \begin{bmatrix} (-1)^{1+1}\begin{vmatrix} 1 & -1 \\ -1 & 1 \end{vmatrix} & (-1)^{1+2}\begin{vmatrix} 2 & -1 \\ 1 & 1 \end{vmatrix} & (-1)^{1+3}\begin{vmatrix} 2 & 1 \\ 1 & -1 \end{vmatrix} \\ (-1)^{2+1}\begin{vmatrix} 1 & 1 \\ -1 & 1 \end{vmatrix} & (-1)^{2+2}\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} & (-1)^{2+3}\begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} \\ (-1)^{3+1}\begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} & (-1)^{3+2}\begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix} & (-1)^{3+3}\begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} \end{bmatrix}$

$$= \begin{bmatrix} +(1-1) & -(2+1) & +(-2-1) \\ -(1+1) & +(1-1) & -(-1-1) \\ +(-1-1) & -(-1-2) & +(1-2) \end{bmatrix}_{(3\times3)}$$

$$= \begin{bmatrix} 0 & -3 & -3 \\ -2 & 0 & 2 \\ -2 & 3 & -1 \end{bmatrix}_{(3\times3)}$$

$$\therefore Adj(A) = \begin{bmatrix} 0 & -3 & -3 \\ -2 & 0 & 2 \\ -2 & 3 & -1 \end{bmatrix}' = \begin{bmatrix} 0 & -2 & -2 \\ -3 & 0 & 3 \\ -3 & 2 & -1 \end{bmatrix}$$

Now putting the values of $|A|$, $Adj(A)$ and C in equation (2), we have -

$$X = \frac{1}{-6} \begin{bmatrix} 0 & -2 & -2 \\ -3 & 0 & 3 \\ -3 & 2 & -1 \end{bmatrix} \begin{bmatrix} 6 \\ 1 \\ 2 \end{bmatrix}$$

$$= \frac{1}{-6} \begin{bmatrix} (0.6 + (-2).1 + (-2).2) \\ ((-3).6 + 0.1 + 3.2) \\ ((-3).6 + 2.1 + (-1).2) \end{bmatrix}$$

$$= \frac{1}{-6} \begin{bmatrix} (0-2-4) \\ (-18+0+6) \\ (-18+2-2) \end{bmatrix} = \frac{1}{-6} \begin{bmatrix} -6 \\ -12 \\ -18 \end{bmatrix}.$$

$$= \begin{bmatrix} \frac{-6}{-6} \\ \frac{-12}{-6} \\ \frac{-18}{-6} \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}.$$

$\therefore x_1 = 1,\ x_2 = 2$ and $x_3 = 3$