

$$\begin{vmatrix} -2 & -3 & 2 \\ 1 & 3 & -2 \\ -1 & -6 & 9 \end{vmatrix} = (-2) \begin{vmatrix} 3 & -2 \\ -6 & 4 \end{vmatrix} - (-3) \begin{vmatrix} 1 & -2 \\ -1 & 4 \end{vmatrix} + 2 \begin{vmatrix} 1 & 3 \\ -1 & -6 \end{vmatrix}$$

$$= (-2)(12 - 12) + 3(4 - 2) + 2(-6 + 3)$$

$$= ((-2) \times 0) + (3 \times 2) + (2 \times (-3))$$

$$= (0 + 6 - 6) = 0$$

Accordingly putting the values of the sub-determinants in equation ①, we have

$$\begin{aligned} |A| &= (2 \times 0) - (5 \times 8) - (3 \times (-12)) + (2 \times 0) \\ &= (0 - 40 + 36 + 0) = -4 \end{aligned}$$

Properties of Determinants: There are some basic features or characteristics of determinants, which are as follows:

1. If all the rows and columns of a determinant are interchanged, the value of the determinant will not change. In other words, the determinant value of a matrix  $A$  has the same or identical value as that of its transpose. Symbolically,  $|A| = |A'|$

If  $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ , then

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{vmatrix} \rightarrow (\text{Transpose of } A)$$

or  $|A| = |A'|$

For example: If  $A = \begin{bmatrix} 2 & 3 \\ 0 & 4 \end{bmatrix}$ , then

determinant value of  $A$  ( $|A|$ ) =  $\begin{vmatrix} 2 & 3 \\ 0 & 4 \end{vmatrix} = (8 - 0) = 8$

Transpose of  $A$  ( $A'$ ) =  $\begin{bmatrix} 2 & 0 \\ 3 & 4 \end{bmatrix}$

Determinant value of transpose of  $A$  ( $|A'|$ ) =  $\begin{vmatrix} 2 & 0 \\ 3 & 4 \end{vmatrix} = (8 - 0) = 8$

Thus,  $|A| = |A'|$

2. If any two adjacent rows (or any two columns) are interchanged, the value of the determinant will not change, but the sign of the determinant will alter. For example:

$$\text{if } |A| = \begin{vmatrix} 2 & 4 & 5 \\ 3 & 2 & 0 \\ 1 & 0 & 1 \end{vmatrix} = 2 \begin{vmatrix} 2 & 0 \\ 0 & 1 \end{vmatrix} - 4 \begin{vmatrix} 3 & 0 \\ 1 & 1 \end{vmatrix} + 5 \begin{vmatrix} 3 & 2 \\ 1 & 0 \end{vmatrix}$$

$$= 2(2-0) - 4(3-0) + 5(0-2)$$

$$= (4-12-10) = -18$$

Now, interchanging the 1st and 2nd rows of  $|A|$ , the determinant looks like

$$\begin{vmatrix} 3 & 2 & 0 \\ 2 & 4 & 5 \\ 1 & 0 & 1 \end{vmatrix} = 3 \begin{vmatrix} 4 & 5 \\ 0 & 1 \end{vmatrix} - 2 \begin{vmatrix} 2 & 5 \\ 1 & 1 \end{vmatrix} + 0 \begin{vmatrix} 2 & 4 \\ 1 & 0 \end{vmatrix}$$

$$= 3(4-0) - 2(2-5) + 0(0-4)$$

$$= 12 + 6 + 0 = +18$$

$$\therefore \begin{vmatrix} 2 & 4 & 5 \\ 3 & 2 & 0 \\ 1 & 0 & 1 \end{vmatrix} = (+) \begin{vmatrix} 3 & 2 & 0 \\ 2 & 4 & 5 \\ 1 & 0 & 1 \end{vmatrix} \leftarrow (\text{by interchanging 1st and 2nd rows})$$

3. If any column (or row) is shifted over any adjacent columns (or row) then value of the determinant will not change, but its sign changes if the jumped columns (or rows) are odd in number and its sign will not change if the jumped columns (or rows) are even in number. For example -

$$\begin{vmatrix} 2 & 2 & 5 & 4 \\ 4 & 1 & 3 & 2 \\ 4 & 1 & 2 & 0 \\ 3 & 2 & 1 & 1 \end{vmatrix} \xrightarrow{\text{2 columns}} = (-)^2 \begin{vmatrix} 2 & 5 & 2 & 4 \\ 1 & 3 & 4 & 2 \\ 1 & 2 & 9 & 0 \\ 2 & 1 & 3 & 1 \end{vmatrix} \quad (\text{The sign of the determinant does not change because jumped columns are 2 in number})$$

(Even)

Again,

$$\begin{vmatrix} 2 & 2 & 5 & 4 \\ 4 & 1 & 3 & 2 \\ 4 & 1 & 2 & 0 \\ 3 & 2 & 1 & 1 \end{vmatrix} \xrightarrow{\text{3 columns}} = (-)^3 \begin{vmatrix} 2 & 5 & 4 & 2 \\ 1 & 3 & 2 & 4 \\ 1 & 2 & 0 & 4 \\ 2 & 1 & 1 & 3 \end{vmatrix} \quad (\text{The sign of the determinant will change because jumped columns are 3 (odd) in number})$$

4. If all the elements of any one row (or column) are multiplied (or divided) by a scalar (say  $\lambda$ ), then the value of the determinant will change by  $\lambda$  times (i.e., multiplied or divided by  $\lambda$ ). For example -

If  $|A| = \begin{vmatrix} 2 & 3 \\ 1 & 0 \end{vmatrix}$ , and the elements of the 1st row are multiplied by  $\lambda$ , then

$$\begin{vmatrix} \lambda \cdot 2 & \lambda \cdot 3 \\ 1 & 0 \end{vmatrix} = (\lambda \cdot 0 - 3 \cdot \lambda) = -3\lambda$$

$= \lambda |A|$ , because

$$\begin{aligned} |A| &= \begin{vmatrix} 2 & 3 \\ 1 & 0 \end{vmatrix} \\ &= 0 - 0 \\ &= -3 \end{aligned}$$

5. If any two rows (or any two columns) of a determinant are identical, then the value of the determinant will be zero. For example -

$$|A| = \begin{vmatrix} 2 & 3 & 1 \\ 2 & 3 & 1 \\ 1 & 3 & 5 \end{vmatrix} = 2 \begin{vmatrix} 3 & 1 \\ 3 & 5 \end{vmatrix} - 3 \begin{vmatrix} 2 & 1 \\ 1 & 5 \end{vmatrix} + 1 \begin{vmatrix} 2 & 3 \\ 1 & 3 \end{vmatrix}$$

$$= 2(15 - 3) - 3(10 - 1) + 1(6 - 3)$$

$$= (2 \times 12) - (3 \times 9) + (1 \times 3)$$

$$= 24 - 27 + 3 = 27 - 27 = 0$$

6. If there is any common factor in the elements of any one row or column, then it can be taken out without affecting the value of the determinant, i.e., this activity will not change the value of the determinant. For example :

$$\begin{vmatrix} 3 & 0 & 1 \\ 6 & 1 & 2 \\ 12 & 2 & 1 \end{vmatrix} = 3 \begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} - 3 \begin{vmatrix} 6 & 2 \\ 12 & 1 \end{vmatrix} + 1 \begin{vmatrix} 6 & 1 \\ 12 & 2 \end{vmatrix} = 3(1 - 4) - 3(6 - 24) + 1(12 - 12) \\ = (3 \times (-3)) - (3 \times (-18)) + (1 \times 0) = (-9 + 54 + 0) = 45$$

Now it is seen that there is a common number, i.e., 3 in the first column and it is taken out.

$$\text{then, } 3 \begin{vmatrix} 1 & 0 & 1 \\ 2 & 1 & 2 \\ 4 & 2 & 1 \end{vmatrix} = 3 \left[ 1 \begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} - 3 \begin{vmatrix} 2 & 2 \\ 4 & 1 \end{vmatrix} + 1 \begin{vmatrix} 2 & 1 \\ 4 & 2 \end{vmatrix} \right] \\ = 3 \left[ 1(1 - 4) - 3(2 - 8) + 1(4 - 4) \right] \\ = 3 \left[ (1 \times (-3)) - (3 \times (-6)) + (1 \times 0) \right] = 3[(-3 + 18 + 0)] \\ = (3 \times 15) = 45$$

Thus, it can be concluded that after taking out the common factor, i.e., 3, value of the determinant does not change.

7. If any multiple of any column (or row) is added or subtracted from any other column (or row), the value of determinant will not change. For example:

$$\begin{vmatrix} 3 & 2 & 1 \\ 1 & 2 & 2 \\ 4 & 3 & 1 \end{vmatrix} = 3 \begin{vmatrix} 2 & 2 \\ 3 & 1 \end{vmatrix} - 2 \begin{vmatrix} 1 & 2 \\ 4 & 1 \end{vmatrix} + 1 \begin{vmatrix} 1 & 2 \\ 4 & 3 \end{vmatrix} = 3(2-6) - 2(1-8) + 1(3-8) \\ = (3 \times (-4)) - (2 \times (-7)) + 1 \times (1 \times (-5)) = (-12 + 14 - 5) = -3$$

Now, multiply the 3rd column by 2 and subtract from column one (1), we have -

$$\begin{vmatrix} (3-2) & 2 & 1 \\ (1-4) & 2 & 2 \\ (4-2) & 3 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 1 \\ -3 & 2 & 2 \\ 2 & 3 & 1 \end{vmatrix} = 1 \begin{vmatrix} 2 & 2 \\ 3 & 1 \end{vmatrix} - 2 \begin{vmatrix} -3 & 2 \\ 2 & 1 \end{vmatrix} + 1 \begin{vmatrix} -3 & 2 \\ 2 & 3 \end{vmatrix} \\ = 1(2-6) - 2((-3)-4) + 1((-9)-4) \\ = (1 \times (-4)) - (2 \times (-7)) + (1 \times (-13)) \\ = (-4 + 14 - 13) = -3$$

Thus, it can be noted that value of the determinant does not alter.

8. If a determinant is of the form

$$\begin{vmatrix} (a_1+b_1) & c_1 & d_1 \\ (a_2+b_2) & c_2 & d_2 \\ (a_3+b_3) & c_3 & d_3 \end{vmatrix}, \text{ then the value of the determinant}$$

is the sum of the following two determinants:

$$\begin{vmatrix} a_1 & c_1 & d_1 \\ a_2 & c_2 & d_2 \\ a_3 & c_3 & d_3 \end{vmatrix} \text{ and } \begin{vmatrix} b_1 & c_1 & d_1 \\ b_2 & c_2 & d_2 \\ b_3 & c_3 & d_3 \end{vmatrix}. \text{ Thus,}$$

$$\begin{vmatrix} (a_1+b_1) & c_1 & d_1 \\ (a_2+b_2) & c_2 & d_2 \\ (a_3+b_3) & c_3 & d_3 \end{vmatrix} = \begin{vmatrix} a_1 & c_1 & d_1 \\ a_2 & c_2 & d_2 \\ a_3 & c_3 & d_3 \end{vmatrix} + \begin{vmatrix} b_1 & c_1 & d_1 \\ b_2 & c_2 & d_2 \\ b_3 & c_3 & d_3 \end{vmatrix}$$

For example:-

$$\begin{vmatrix} 3 & 2 & 1 \\ 6 & 4 & 2 \\ 3 & 2 & 1 \end{vmatrix} = \begin{vmatrix} (2+1) & 2 & 1 \\ (4+2) & 4 & 2 \\ (2+1) & 2 & 1 \end{vmatrix} = \begin{vmatrix} 2 & 2 & 1 \\ 4 & 4 & 2 \\ 2 & 2 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 1 \end{vmatrix}, \text{ because}$$

$$\begin{vmatrix} 3 & 2 & 1 \\ 6 & 4 & 2 \\ 3 & 2 & 1 \end{vmatrix} = \begin{vmatrix} (3 \times 1) & 2 & 1 \\ (3 \times 2) & 4 & 2 \\ (3 \times 1) & 2 & 1 \end{vmatrix} = 0, \text{ because the 1st column of the determinant is multiple of the 3rd column. (Property 8), hence value of the determinant is zero.}$$

Now,  $\begin{vmatrix} 2 & 2 & 1 \\ 4 & 4 & 2 \\ 2 & 2 & 1 \end{vmatrix} = 0$ , because, the 1st and 2nd column of the determinant are identical, hence value of it will be zero. (Property 5).

Similarly,  $\begin{vmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 1 \end{vmatrix} = 0$ , because the 1st and 3rd columns are identical, hence value of the determinant will be zero (Property 5).

8. If any one row (or any one column) of a determinant is a multiple of any other row (or column), the value of <sup>the determinant</sup> it will be zero. For example -

$$|A| = \begin{vmatrix} 3 & 2 & 1 \\ 6 & 4 & 2 \\ 3 & 2 & 1 \end{vmatrix}; \text{ where 1st column is twice (2) of the 3rd column}$$

$$\begin{aligned} \therefore |A| &= \begin{vmatrix} 3 & 2 & 1 \\ 6 & 4 & 2 \\ 3 & 2 & 1 \end{vmatrix} = 3 \begin{vmatrix} 4 & 2 \\ 2 & 1 \end{vmatrix} - 2 \begin{vmatrix} 6 & 2 \\ 3 & 1 \end{vmatrix} + 1 \begin{vmatrix} 6 & 4 \\ 3 & 2 \end{vmatrix} \\ &= 3(4-4) - 2(6-6) + 1(12-12) \\ &= (3 \times 0) - (2 \times 0) + (1 \times 0) \\ &= (0 - 0 + 0) = 0 \end{aligned}$$

9. If a determinant is expanded by its alien co-factors, the value of the determinant will be zero. In other words, if any one row (or column) is considered for expansion of a determinant by using co-factors of some other row (or column), then the value of the determinant will be zero. For example

$$\begin{aligned}
 |A| &= \begin{vmatrix} 3 & 2 & 1 \\ 1 & 2 & 2 \\ 4 & 3 & 1 \end{vmatrix} = 3 \begin{vmatrix} 2 & 2 \\ 3 & 1 \end{vmatrix} - 2 \begin{vmatrix} 1 & 2 \\ 4 & 1 \end{vmatrix} + 1 \begin{vmatrix} 1 & 2 \\ 4 & 3 \end{vmatrix} = 3(2-6) - 2(1-8) + 1(3-8) \\
 &= (3 \times (-4)) - (2 \times (-7)) + (1 \times (-5)) = (-12 + 14 - 5) = -3
 \end{aligned}$$

But, if the determinant is expanded by considering the 1st row elements using co-factors of 3rd row, then

$$\begin{aligned}
 |A| &= 3 \begin{vmatrix} 0 & 1 \\ 2 & 2 \end{vmatrix} - 2 \begin{vmatrix} 3 & 1 \\ 1 & 2 \end{vmatrix} + 1 \begin{vmatrix} 3 & 2 \\ 1 & 2 \end{vmatrix} = 3(4-2) - 2(6-1) + 1(6-2) \\
 &= (3 \times 2) - (2 \times 5) + (1 \times 4) \\
 &= (6 - 10 + 4) = 0
 \end{aligned}$$